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Cooperative Classes of Finite Sets in One and More Dimensions*

THEODORE S. MOTZKIN

University of California, Los Angeles, and Rockefeller University, New York

ABSTRACT

For $x_1 < x_2 < x_3$ let $\sigma = (x_1 - x_2)/(x_2 - x_3)$, $\rho = \max(\sigma, 1/\sigma)$. For $\alpha > 1$, the class C_α of 3-sets with $\rho < \alpha$ is non-cooperative: for $n \geqslant \alpha \log 4 + 2$, no n-set has all its 3-subsets in C_α. For $n \geqslant$ the Ramsey number $N([\alpha \log 4 + 3], q, q; 3)$, every n-set has a q-subset none of whose 3-subsets is in C_α. Likewise for $q > d \geqslant 1$, $\epsilon > 0$, $\alpha > 1$ there is a bound B such that every general n-ad, $n \geqslant B$, in d-space has a comonotone q-sub-ad whose 2-sub-ads form with a certain ray R an angle $< \epsilon$ and whose 3-sub-ads, orthogonally projected on R, are not in C_α. Finally, for every general $N(7, 7; 4)$-set S in 3-space there exists a knotted polygon whose vertices belong to S.

1. INTRODUCTION

The influence on mathematics of its two neighbors, physics and logic, is sometimes opposite or, at least, complementary. Whereas the entropy theorems of probability theory and mathematical physics imply that, in a large universe, disorder is probable, certain combinatorial theorems show that complete disorder is impossible. Already the "pigeon-hole principle" —out of $t + 1$ objects of t kinds, at least two must be of the same kind— can be interpreted as saying that, while no 2-set (set of two objects) in the given $(t + 1)$-set needs to be composed of objects of different kind (however probable that may be), at least one 2-set of objects of the same kind must occur. A sophisticated generalization of this principle was initiated in [7] by the mathematical logician Ramsey. Ramsey's theorem ([8, p. 39] states:

For any positive integers $q_1, ..., q_t$, $r (r \leqslant \min q_j)$, there is a number N with the following property. If the union of the sets $A_1, ..., A_t$ of r-sets

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contains all r-subsets of an N-set then there is a j and a q_j-set all of whose r-subsets are members of A_j.

The smallest such N will be denoted by $N(q_1, ..., q_t; r)$.

Using this theorem we establish three types of orderliness whose (in fact—see Section 8—even simultaneous) occurrence can be guaranteed in every sufficiently large point set in real d-space: comonotonicity (Section 6), almost-straightness (Section 7), geometric almost-degeneracy (Section 3).

2. A NON-COOPERATIVE CLASS

Given a 3-set (set of 3 points) $\{x_1, x_2, x_3\}$, where $x_1 < x_2 < x_3$, in real affine 1-space, we call

$$\max((x_1 - x_2)/(x_3 - x_3),(x_3 - x_2)/(x_2 - x_1))$$

its *ratio* ρ; then $\rho \geqslant 1$. We have

THEOREM 1. *The largest α such that, among the 3-subsets of an arbitrary n-set in real affine 1-space, at least one has a ratio $\rho \geqslant \alpha$, is*

$$\alpha_n = \beta_n/(1 - \beta_n), \qquad \textit{where} \quad \begin{cases} 2\beta^{(n-1)/2} = 1 & \textit{for odd } n, \\ \beta^{n/2-1} + \beta^{n/2} = 1 & \textit{for even } n. \end{cases}$$

PROOF: For a given n-set $\{x_1, ..., x_n\}$, $x_1 < \cdots < x_n$, let S be the set of all those 3-subsets for which ρ is largest, say $= \rho_0$. Then the members of S are of the form $\{x_1, x_k, x_{k+1}\}$, $x_1 + x_n \leqslant x_k + x_{k+1}$, or $\{x_k, x_{k+1}, x_n\}$, $x_1 + x_n \geqslant x_k + x_{k+1}$. If, of the two 2-sets $\{x_{k-1}, x_k\}$ and $\{x_k, x_{k+1}\}$, only one is part of a member of S then, after a slight change of x_k, neither will be part of a member of S; by continuing in the same way, S becomes empty and ρ_0 decreases. In the limit we are led to an n-set such that each $\{x_k, x_{k+1}\}$ is part of a member of S. Then there is a j such that S consists of all (x_1, x_k, x_{k+1}), $k = j, ..., n-1$ and all (x_k, x_{k+1}, x_n), $k = 1, ..., j-1$ and possibly $k = j$. Letting $x_1 = 0$, $x_n = 1$, $\sigma_0 = \rho_0/(\rho_0 + 1)$ we have $x_{n-1} = \sigma_0, ..., x_j = \sigma_0^{n-j}$ and $x_2 = 1 - \sigma_0, ..., x_j = 1 - \sigma_0^{j-1}$. Here $n - j$ is the smallest number such that $\sigma_0^{n-j} \leqslant \frac{1}{2}$. If $\sigma_0^{n-j} = \frac{1}{2}$ then $\sigma_0^{j-1} = \frac{1}{2}$, $j = (n-1)/2$. If $\sigma_0^{n-j} < \frac{1}{2}$ then $\sigma_0^{n-j} + \sigma_0^{n-j-1} > 1$; but $\sigma_0^{j-1} = 1 - \sigma_0^{n-j}$ cannot lie in the open interval $(\sigma_0^{n-j}, \sigma_0^{n-j-1})$, hence $\sigma_0^{j-1} = \sigma_0^{n-j-1}$, $j = n/2$.

It is not hard to verify $\alpha_n = (n-1)/\log 4 - \frac{1}{2} + O(1/n)$.

There follows

THEOREM 2. *For* $\alpha \geqslant 1$ *there exists a bound* $N(\alpha)$ *such that every n-set,* $n \geqslant N(\alpha)$, *contains a* 3-*set whose ratio is* $> \alpha$; *indeed,*

$$N(\alpha) = \alpha \log 4 + 1 + \log 2 + O(1/\alpha) + \vartheta, 0 < \vartheta \leqslant 1.$$

The class of 3-sets with a ratio $\leqslant \alpha$ is thus not *cooperative*, i.e., one cannot form arbitrarily large sets all of whose 3-sets belong to this class.

3. Geometrically Almost Degenerate Sets

On the other hand it is easy to form, for every q, q-sets

$$\{x_1, ..., x_q\}, x_1 < \cdots < x_q,$$

all of whose 3-subsets have a ratio $> \alpha$; e.g., it suffices to have

$$(x_{k+1} - x_k)/(x_q - x_{k+1}) > \alpha \text{ for } k = 1, ..., q - 2.$$

If this latter condition is fulfilled the set is called *α-decreasingly geometrically almost degenerate*, or *α-decreasing*, while $\{-x_1, ..., -x_q\}$ is *α-increasing* ("geometric" recalls the situation where the ratios equal α, "almost degenerate" the coalescence of $x_2, ..., x_q$ for $\alpha \to \infty$). We now prove

THEOREM 3. *For every real* $\alpha \geqslant 1$ *and every integer* $q \geqslant 3$ *there exists a bound* $N(q, \alpha)$ *such that every n-set,* $n > N(q, \alpha)$, *has an α-decreasing or α-increasing q-subset.*

PROOF: We establish $N(q, \alpha) \leqslant N(N(\alpha), q, q; 3)$ by using Theorem 2 and Ramsey's theorem, defining A_j as the set of all 3-subsets

$$\{x_1, x_2, x_3\}, x_1 < x_2 < x_3,$$

of the given n-set

for $j = 1$: whose ratio is $\leqslant \alpha$;
for $j = 2$: for which $(x_1 - x_2)/(x_2 - x_3) > \alpha$;
for $j = 3$: for which $(x_3 - x_2)/(x_2 - x_1) > \alpha$.

Note that the 4-sets all of whose 3-subsets have a ratio $> \alpha$ but which are neither α-decreasing nor α-increasing form a non-cooperative class. However, the 5-sets all of whose 4-subsets are α-decreasing or α-increasing are themselves α-decreasing or α-increasing. Had we not divided the 3-sets into those with a ratio $> \alpha$ and those with a ratio $\leqslant \alpha$, but into (1) the latter, (2) the α-decreasing, and (3) the α-increasing 3-sets, then already those 4-sets whose 3-subsets are all of type (2), or all of type (3), would be of the same type, without further splitting off a non-cooperative class.

4. Convex Finite Sets

In affine d-space, a point is represented by its d coordinates, preceded by a 1, and an n-ad (sequence of n points) by the n by $d + 1$ matrix formed by the corresponding rows, called the *augmented matrix*. An n-set, $n \geqslant d + 1$, is *general* if no $(d + 1)$-subset is on a hyperplane.

In real affine d-space, an n-set is *convex* if its points are the vertices of a convex d-polyhedron. Every general $(d + 1)$-set is convex, and a general $(d + 2)$-set is convex unless, in the linear relation between the rows of the augmented matrix, all coefficients but one have the same sign. Using this remark we derive

Theorem 4. *The number of non-convex $(d + 2)$-subsets among the $d + 3$ $(d + 2)$-subsets of a general $(d + 3)$-set is* 4 *for* $d = 1$, 0, 2 *or* 4 *for* $d \geqslant 2$.

Proof: Let $\lambda_1, \ldots, \lambda_{d+3}$ and $\mu_1, \ldots, \mu_{d+3}$ be the coefficients of two independent linear relations between the rows of the augmented matrix of the $(d + 3)$-set; every linear relation between the rows is then a linear combination of these two relations. Consider the $d + 3$ points $(\lambda_1, \mu_1), \ldots, (\lambda_{d+3}, \mu_{d+3})$ in 2-space; the distances, from the straight line l_j through the origin and (λ_j, μ_j), of the other $d + 2$ points are the coefficients of the linear relation belonging to that $(d + 2)$-set. This set is non-convex if and only if the sum of the angles, seen from the origin, between (λ_j, μ_j) and its two successors, in one of the cyclic senses, exceeds π. Each excessive angle pair corresponds to two non-convex $(d + 2)$-sets; and the number of such pairs is obviously 1 for $d + 3 = 4$, 0, 1 or 2 for $d + 3 \geqslant 5$.

We see that for $d \geqslant 2$ the class of non-convex general $(d + 2)$-sets is non-cooperative, which by Ramsey's theorem guarantees convex q-subsets for n-sets with sufficiently large n. But we shall (as indicated in [4]) establish non-cooperativity of a, for $d \geqslant 4$, larger class, that of not mostconvex general $(d + 2)$-sets.

5. Mostconvex Subsets

If the maximum number of coefficients with equal sign in the linear relation between the rows of the augmented matrix of a general $(d + 2)$-set is $d+2-k$ the set is called $(k-1)$-*convex*; 0-convex is the same as non-convex. A $[d/2]$-convex set is also called *mostconvex*; for $d \geqslant 2$, a mostconvex set is convex. We prove

THEOREM 5. *The number of mostconvex $(d+2)$-subsets among the $d+3$ $(d+2)$-subsets of a general $(d+3)$-set is at least* 1, *and for odd d at least* 2.

PROOF: For an open half-plane bounded by some l_j, in the proof of Theorem 4, let κ be the number of those among the $d+2$ points that are in the half-plane. While the half-plane assumes the $2(d+3)$ possible positions in cyclic order, κ varies "near-continuously," namely each time increasing by 1, 0, or -1. If κ belongs to a certain half-plane, $d+2-\kappa$ belongs to the opposite half-plane; choosing, if possible, κ and $d+2-\kappa \neq [d/2]+1$, and turning cyclically from one to the other, there must, because of the near-continuity, occur at least one intermediate value $[d/2]+1$, and at least one $d+1-[d/2]$.

We can now prove

THEOREM 6. *For every two integers $d \geqslant 1$ and $q \geqslant d+2$ there exists a bound $M(q, d)$ such that every general n-set, $n \geqslant M(q, d)$, in real affine d-space has a q-subset all of whose $(d+2)$-subsets are mostconvex.*

PROOF: Theorem 5 and Ramsey's theorem imply

$$M(q, d) \leqslant N(d+3, q; d+2).$$

All $(d+2)$-subsets of a $(d+3)$-set are mostconvex if and only if κ, in the proof of Theorem 5, is $[d/2]+1$ or $d+1-[d/2]$ for every half-plane. For even d this means that the rays from the origin on the lines l_j alternatingly contain, and do not contain, one of the points $\{\lambda_j, \mu_j\}$; this we write symbolically $+-+-\cdots$. For odd d the $++$ and the $--$ must occur alternatingly, with strings of alternating $+$ and $-$ between them. For $d=1$ the symbol must be $++-+--+-$ or, writing one half only, $++-+$. For $d=3$ the half-symbol is $++--++$ or $++-+-+$; indicating half the distance between each $++$ or $--$ and the next one, we can instead write 111 or 3. For any odd d, the latter kind of symbol is a dihedral partition of $(d+3)/2$ into an odd number of parts; for $d = 1, 3, 5, 7, 9, 11,\ldots$ there are $1, 2, 2, 4, 5, 9,\ldots$ such partitions (cf. [1]).

For $d \geqslant 2$, the symbol $++++-\cdots$ (alternating from here on) represents a general $(d+3)$-set for which the number of mostconvex $(d+2)$-subsets attains the lower bound given in Theorem 5. (For $d=1$ the number is necessarily 4, as all 3-sets are mostconvex.)

6. Comonotone and Knotted Finite Sets

A $(d+1)$-ad is *positive* or *negative* if the determinant of its augmented matrix is positive or negative. An n-ad is

general if each $(d+1)$-sub-ad is positive or negative;
positive (*negative, non-negative, nonpositive*) if each $(d+1)$-sub-ad is positive (negative, non-negative, non-positive);
comonotone [2, 3] if it is positive or negative.

(For $d = 1$ comonotone becomes monotone.) Each of these six properties of an n-ad is shared by all its sub-ads.

On the other hand, non-comonotonicity is non-cooperative, as far as general n-ads are concerned. In fact, we have

Theorem 7. *For every two integers $d \geqslant 1$ and $q \geqslant d+1$ there exists a bound $C(q, d)$ such that every n-ad, $n \geqslant C(q, d)$, in real affine d-space has a positive or nonpositive q-sub-ad.*

Proof: Ramsey's theorem implies $C(q, d) \leqslant N(q, q; d+1)$. (For related results see [5, §4].)

If the n-ad is general then the q-sub-ad is necessarily comonotone. (This is one way of showing that comonotone q-ads exist.)

Likewise we obtain:

Theorem 8. *For every two integers $d \geqslant 1$ and $q \geqslant d+1$ there exists a bound $H(q, d)$ such that every n-ad, $n \geqslant H(q, d)$, in real affine d-space has a q-sub-ad that is either comonotone or on a hyperplane.*

Proof: Ramsey's theorem implies $H(q, d) \leqslant N(q, q, q; d+1)$.

A $(d+2)$-set is mostconvex if and only if it has a comonotone $(d+2)$-ad.

For even d the symbol (Section 5) of a $(d+3)$-set with a comonotone $(d+3)$-ad is $+-+-\cdots$, and every cyclic permute of the ad is also comonotone. For odd d, $+-+-\cdots$ is half the symbol, and the numerical symbol is $(d+3)/2$.

Thus for odd $d \geqslant 3$, the class of $(d+3)$-sets all of whose $(d+2)$-subsets are mostconvex splits into the cooperative class A of those $(d+3)$-sets that have a comonotone $(d+3)$-ad, and into a non-cooperative complementary class A'.

An n-set in real affine 3-space is *knotted* if there exists a knotted polygon whose vertices belong to the n-set. It is not hard to see that the only knotted n-sets with $n \leqslant 6$ are the 6-sets of class A'. The points of such a 6-set are the vertices of a polyhedron with the combinatorial face structure of the octahedron, and they are also the vertices of 8 knotted 6-gons.

In answer to a question of [6] we prove:

THEOREM 9. *There exists a bound K such that every general n-set, $n \geqslant K$, is knotted.*

PROOF. By Theorem 7 we have $K \leqslant N(7, 7; 4)$ if every comonotone 7-set is knotted. Now the segments connecting the points of a comonotone 7-set can be represented by those connecting the points $x_1, ..., x_7$ of a slightly perturbated convex 7-set in 2-space in such a way that the segment $x_j x_k$, $j < k$, passes above $x_l x_m$, $l < m$, if $j < l$. It is then easy to confirm that the polygon $x_1 x_3 x_5 x_7 x_2 x_4 x_6$ is knotted.

7. Almost Straight Ang Almost Flat Sets

A set is called *ϵ-straight* if there exists a straight line whose angle with every straight line through two points of the set is $< \epsilon$. An ad is called *ϵ-straight* if there exists a ray ("main ray") whose angle with every direction defined by a 2-sub-ad (whose points do not coincide) is $< \epsilon$. Obviously the set of an ϵ-straight ad is ϵ-straight, and an ϵ-straight n-set, $n \geqslant 2$, has at least two ϵ-straight ads. We prove

THEOREM 10. *For every two integers $d \geqslant 1$ and $q \geqslant 2$ and for every $\epsilon > 0$ there exists a bound $S(q, d, \epsilon)$ such that every n-set, $n \geqslant S(q, d, \epsilon)$, in Euclidean d-space has an ϵ-straight q-subset.*

PROOF: Let $\lambda = \lambda_{d,\epsilon}$ be the smallest number of open ϵ-balls which, together with their antipodal balls, cover the unit sphere $|x| = 1$ of d-space. Then Ramsey's theorem implies $S(q, d, \epsilon) \leqslant N(q, ..., q(\lambda \text{ times}); 2)$.

The corresponding statement holds also for ads. In the proof, omit the words "together with their antipodal balls."

Similarly we can, for $d' \leqslant d$, require ϵ, *d'-flatness* of subsets or sub-ads, i.e., angles smaller than ϵ between a d'-plane (without or with orientation) and the d'-planes connecting the points of the subset or sub-ad.

8. Combination of Properties

If, for $j = 1$ and $j = 2$ and every positive integer q, every n-set (within a given set of sets), $n \geqslant C_j(q)$, has a q-subset all of whose r_j-subsets are of class A_j then clearly every n-set, $n \geqslant C_1(C_2(q))$, has a q-subset all of whose

r_1-subsets are of class A_1 and all of whose r_2-subsets are of class A_2. Using this argument twice, we obtain from Theorems 3, 7, and 10:

THEOREM 11. *For every two integers $d \geqslant 1$ and $q \geqslant d + 1$, and for every $\epsilon > 0$ and $\alpha \geqslant 1$ there exists a bound $B(q, d, \epsilon, \alpha)$ such that every general n-ad, $n \geqslant B(q, d, \epsilon, \alpha)$, in Euclidean d-space has a comonotone and ϵ-straight q-sub-ad whose orthogonal projection on the main ray is α-increasing or α-decreasing.*

For $q \geqslant 3$ and small ϵ, a positive and a negative ϵ-straight q-ad with α-decreasing projection cannot belong to the same q-set.

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